HEAT TRANSFER IN THE THERMALLY STABILIZED SECTIONS OF CHANNELS WITH A LAMINAR NON-NEWTONIAN FLUID FLOW

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An approximate method is described for solving the problem of heat transfer for laminar non-newtonian fluid flow in the thermally stabilized sections of a plane channel and a circular tube under conditions of constant wall temperature.

Non-newtonian fluids are met with in various technical applications. The basic hydrodynamic feature of the behavior of these fluids is the fact that their motion cannot be described in accordance with the newtonian hypothesis of a linear relation between stress and rate of strain

$$-\frac{du}{dr}=\frac{1}{\eta}\tau. \tag{1}$$

For non-newtonian fluids the relation between stress and shear rate is nonlinear.

Various rheological laws of flow of non-newtonian fluids are known. The most widely used of these is the so-called power law, which may be written in the empirical form [1]

$$-\frac{du}{dr}=K\tau^n, \tag{2}$$

where K and n are individual rheological constants of the substance; if K depends strongly on temperature, the quantity n varies to a much lesser degree and can often be considered constant. When n is 1 we have an ordinary newtonian fluid.

In spite of the fact that over a wide range of values of shear rate du/dr the simple rheological power law (2) does not, as a rule, describe the flow of real non-newtonian fluids, the law is applicable to many non-newtonian fluids over a certain range, e.g., to polymer solutions and melts, and it is widely used in various technical calculations [2].

The present paper obtains an approximate theoretical solution to the problem of heat transfer in the laminar flow of a non-newtonian fluid obeying law (2) in the thermally stabilized sections of a plane channel (flow between two parallel infinite planes) and a circular tube under conditions of constant wall temperature.

The solution has been obtained on the following assumptions: the stream is hydrodynamically stabilized; heat transmission in the axial direction is insignificant; the physical properties of the fluid are constant at any point in the flow; the heat resulting from energy dissipation is negligibly small.

In theory the temperature profile, and therefore the heat transfer relations, may be found from the solution of the differential energy equation:

for a plane channel

$$u\frac{\partial t}{\partial x}=a\frac{\partial^2 t}{dh^2}, \tag{3}$$

for a circular tube

$$u\frac{\partial t}{\partial x}=a\left(\frac{\partial^2 t}{\partial r^2}+\frac{1}{r}\frac{\partial t}{\partial r}\right) \tag{4}$$

with appropriate boundary conditions.

For stabilized flow of a non-newtonian fluid, by integrating (2) and using the boundary condition that when h = H or r = R the velocity u of the liquid is 0, we may establish the relation between the flow velocity and the mean velocity $\bar{u}$ over the section [2]:

for a plane channel

$$u(h)=\frac{n+2}{n+1}\bar{u}\left[1-\left(\frac{h}{H}\right)^{n+1}\right], \tag{5}$$

for a circular tube

$$u(r)=\frac{n+3}{n+1}\bar{u}\left[1-\left(\frac{r}{R}\right)^{n+1}\right]. \tag{6}$$

Solution of differential equations (3) and (4) for arbitrary values of n is very difficult. An approximate solution of this equation is given below, using integral relations in the form of a heat content balance for the plane channel and the circular tube.

Plane channel. The heat transfer coefficient may be found from the well-known expression

$$\alpha=q_w/(t_w-t_m). \tag{7}$$

The mean fluid temperature is found from the equation

$$t_m=\frac{1}{H\bar{u}}\int_0^H u(h)\,t(h)\,dh, \tag{8}$$

or, in dimensionless form,

$$\frac{t_m}{t_w}=\int_0^1\frac{u(h)}{\bar{u}}\,\frac{t(h)}{t_w}\,d\left(\frac{h}{H}\right). \tag{9}$$

Taking into account that

$$q=-\lambda\frac{\partial t}{\partial h}, \tag{10}$$

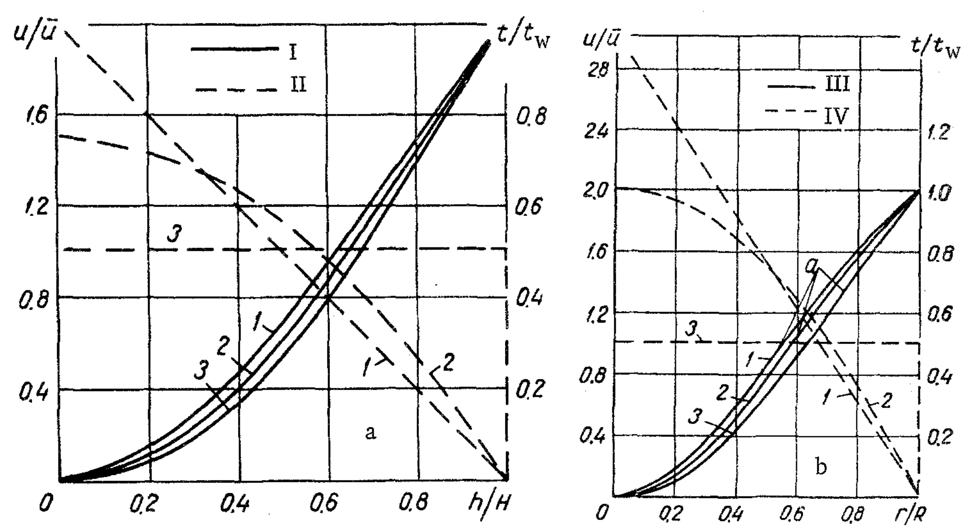

Fig. 1. Distribution of velocity and temperature over a) plane and b) circular channels in the stabilized section of a laminar flow of non-newtonian fluid: I) the relation $t(h/H)/t_W = f(h/H)$; II) $u(h/H)/\bar{u} = f(h/H)$; III) $t(r/R)/t_W = f(r/R)$; IV) $u(r/R)/\bar{u} = = f(r/R)$; 1) when n = 0; 2) 1; 3) ∞.

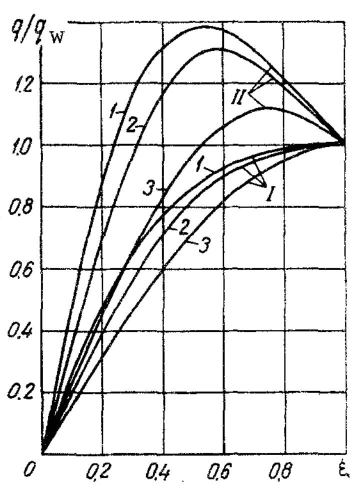

Fig. 2. Dependence of heat flux q/q_W on the dimensionless coordinate ξ: I) for a plane channel; II) for a circular tube; 1) with n = 0; 2) 1; 3) ∞.

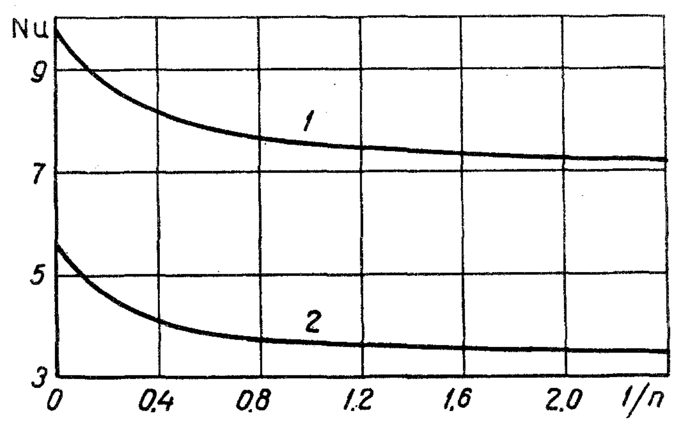

Fig. 3. Nusselt number for a thermally stabilized laminar stream of non-newtonian fluid in 1) a plane channel, and 2) a circular tube.

and integrating this expression, we obtain

$$t_w = \frac{q_w H}{\lambda} \int_0^1 \frac{q}{q_w} d\left(\frac{h}{H}\right). \qquad (11)$$

Substituting (9) and (11) into (7), and bringing it into dimensionless form, we write

$$\mathrm{Nu} = \frac{2\alpha H}{\lambda} = 4 \Bigg/ \left(1 - \frac{t_m}{t_w}\right) \int_0^1 \frac{q}{q_w} d\left(\frac{h}{H}\right). \qquad (12)$$

It follows from (9) and (11) that the temperature profiles for t_m and t_w may be found, if we know the nature of the variation of q/q_w over the height of the channel. This may be found from the equations of heat balance and the condition of similarity of temperature profiles for the stabilized section, i.e.,

$$f_1(t) = kf_2(t).$$

These conditions, along with the Fourier equation, give the following system of equations, written in dimensionless form:

$$\frac{Q}{Q_w} = \frac{q}{q_w}, \qquad (13)$$

$$\frac{Q}{Q_w} = \int_0^\xi \frac{u(\xi)}{\bar{u}}\left[1 - \frac{t(\xi)}{t_w}\right] d\xi \Bigg/ \int_0^1 \frac{u(\xi)}{\bar{u}}\left[1 - \frac{t(\xi)}{t_w}\right] d\xi, \qquad (14)$$

$$\frac{t(\xi)}{t_w} = \int_0^\xi \frac{q}{q_w} d\xi \Bigg/ \int_0^1 \frac{q}{q_w} d\xi,$$

$$\frac{u(\xi)}{\bar{u}} = \frac{n+2}{n+1}(1 - \xi^{n+1}). \qquad (15)$$

This system of equations is solved by a successive approximation method. Putting $q/q_w = 1$ in the first approximation and calculating $t(\xi)/t_w$ according to (15), we find the value of q/q_w in the second approximation from (14). The successive approximation process is very simple, and the solution quickly converges. The difference between the third and fourth approximations to the relation $q/q_w = f(\xi)$ is about 0.5%. It is therefore sufficient to limit the calculation to the third approximation.

Following solution of the problem in general form, the formulas for temperature profiles and Nusselt number according to the third approximation have the form

$$\frac{t}{t_w} = \left\{\frac{a\xi^2}{2} - \frac{\xi^4}{24} + \frac{\xi^5}{120} - \frac{a\xi^{n+3}}{(n+2)(n+3)} + \left[\frac{1}{2(n+4)(n+5)} + \frac{1}{(n+2)(n+3)(n+4)(n+5)}\right]\xi^{n+5} - \left[\frac{1}{6(n+5)(n+6)} + \frac{1}{(n+3)(n+4)(n+5)(n+6)}\right]\xi^{n+6} - \frac{\xi^{2n+6}}{(n+2)(n+3)(2n+5)(2n+6)} + \frac{\xi^{2n+7}}{(n+3)(n+4)(2n+6)(2n+7)}\right\} \Bigg/ c, \qquad (16)$$

$$\mathrm{Nu} = 4b \Bigg/ \left(c - \frac{n+2}{n+1} d\right). \qquad (17)$$

Here

$$\xi = h/H; \quad a = \frac{1}{3} - \frac{1}{(n+2)(n+3)} + \frac{1}{(n+3)(n+4)};$$

$$b = \frac{a(n+1)}{n+2} - \frac{n+1}{6(n+4)} + \frac{n+1}{24(n+5)} + \frac{n+1}{(n+2)(n+3)(n+4)(2n+5)} - \frac{n+1}{(n+3)(n+4)(n+5)(2n+6)};$$

$$c = \frac{a(n+1)(n+4)}{2(n+2)(n+3)} - \frac{(n+1)(n+8)}{24(n+4)(n+5)} + \frac{(n+1)(n+10)}{120(n+5)(n+6)} + \frac{(n+1)(3n+10)}{(n+2)(n+3)(n+4)(n+5)(2n+5)(2n+6)} - \frac{(n+1)(3n+12)}{(n+3)(n+4)(n+5)(n+6)(2n+6)(2n+7)};$$

$$d = a\left[\frac{n+1}{6(n+4)} - \frac{n+1}{(n+2)(n+3)(n+4)(2n+5)}\right] - \frac{n+1}{120(n+6)} + \frac{n+1}{720(n+7)} + \frac{n+1}{(n+2)(n+3)(n+4)(n+5)(n+6)(2n+7)} - \frac{n+1}{(n+3)(n+4)(n+5)(n+6)(n+7)(2n+8)} + \frac{n+1}{2(n+4)(n+5)(n+6)(2n+7)} - \frac{n+1}{6(n+5)(n+6)(n+7)(2n+8)} - \frac{n+1}{(n+2)(n+3)(2n+5)(2n+6)(2n+7)(3n+8)} + \frac{n+1}{(n+3)(n+4)(2n+6)(2n+7)(2n+8)(3n+9)}.$$

Circular channel. This problem is similarly treated. In this case the initial system of equations takes the form

$$\frac{Q}{Q_w} = \frac{q}{q_w}\xi, \qquad (18)$$

$$\frac{\tilde{Q}}{Q_w} = \tag{19}$$

$$= \int_0^\xi \frac{u(\xi)}{\bar{u}}\left[1-\frac{t(\xi)}{t_w}\right]\xi d\xi \Big/ \int_0^1 \frac{u(\xi)}{\bar{u}}\left[1-\frac{t(\xi)}{t_w}\right]\xi d\xi,$$

$$\frac{t(\xi)}{t_w} = \int_0^\xi \frac{q}{q_w} d\xi \Big/ \int_0^1 \frac{q}{q_w} d\xi,$$

$$\frac{u(\xi)}{\bar{u}} = \frac{n+3}{n+1}\left[1-\xi^{n+1}\right]. \tag{20}$$

Values of the mean temperature and Nu number are determined from the formulas

$$\frac{t_m}{t_w} = 2\int_0^1 \frac{u(\xi)}{\bar{u}}\frac{t(\xi)}{t_w}\xi d\xi, \tag{21}$$

$$\mathrm{Nu} = 2 \Big/ \left[1-\frac{t_m}{t_w}\right]\int_0^1 \frac{q}{q_w} d\xi. \tag{22}$$

Following solution in general form, in the case of a circular channel the formulas for temperature profile and Nusselt number, in the third approximation, have the form

$$\frac{t(\xi)}{t_w} = \left\{\frac{a\xi^2}{4} - \frac{\xi^4}{64} + \frac{\xi^5}{225} - \frac{a\xi^{n+3}}{(n+3)^2} + \right.$$
$$+\left[\frac{1}{4(n+5)^2} + \frac{1}{(n+3)^2(n+5)^2}\right]\xi^{n+5} -$$
$$-\left[\frac{1}{9(n+6)^2} + \frac{1}{(n+4)^2(n+6)^2}\right]\xi^{n+6} -$$
$$\left. - \frac{\xi^{2n+6}}{4(n+3)^4} + \frac{\xi^{2n+7}}{(n+4)^2(2n+7)^2}\right\} \Big/ c, \tag{23}$$

$$\mathrm{Nu} = 2b \Big/ \left(c - 2\frac{n+3}{n+1}d\right). \tag{24}$$

Here

$$\xi = \frac{r}{R};\ a = \frac{5}{36} - \frac{1}{(n+3)^2} + \frac{1}{(n+4)^2};$$

$$b = \frac{a(n+1)}{2(n+3)} - \frac{n+1}{16(n+5)} + \frac{n+1}{45(n+6)} +$$
$$+ \frac{n+1}{2(n+3)^3(n+5)} - \frac{n+1}{(n+4)^2(n+6)(2n+7)};$$

$$c = \frac{a(n+1)(n+5)}{4(n+3)^2} - \frac{(n+1)(n+9)}{64(n+5)^2} +$$
$$+ \frac{(n+1)(n+11)}{225(n+6)^2} + \frac{(n+1)(3n+11)}{4(n+3)^4(n+5)^2} -$$
$$- \frac{(n+1)(3n+13)}{(n+4)^2(n+6)^2(2n+7)^2};$$

$$d = a\left[\frac{n+1}{16(n+5)} - \frac{n+1}{2(n+3)^3(n+5)}\right] -$$
$$- \frac{n+1}{384(n+7)} + \frac{n+1}{1575(n+8)} +$$
$$+ \frac{n+1}{4(n+5)^2(n+7)(2n+8)} -$$
$$- \frac{n+1}{9(n+6)^2(n+8)(2n+9)} +$$
$$+ \frac{n+1}{(n+3)^2(n+5)^2(n+7)(2n+8)} -$$
$$- \frac{n+1}{4(n+3)^4(2n+8)(3n+9)} -$$
$$- \frac{n+1}{(n+4)^2(n+6)^2(n+8)(2n+9)} +$$
$$+ \frac{n+1}{(n+4)^2(2n+7)^2(2n+9)(3n+10)}.$$

The results of the velocity and temperature profile calculations according to (5), (6), (16), and (23) are shown in Fig. 1. Comparison of the temperature and velocity profiles shows that the velocity profiles for various values of the exponent n differ appreciably, while the temperature profiles scarcely change with change of n from 0 to ∞. This is very important in analytical examination of the influence of variation of physical properties of the fluid on heat transfer and resistance.

Figure 2 shows the variation of heat flux as a function of the relative coordinate ξ. Because of reduction of the area of heat transfer to the tube axis, the relation $q/q_w = f(r/R)$ has a maximum, corresponding to the bend in the temperature profile (the point a in Fig. 1b). For a plane channel the relation $q/q_w = f(h/H)$ has no maxima.

Figure 3 shows values of stabilized Nusselt number for various values of the rheological constant n, calculated from (17) and (24).

For convenience in calculating the heat transfer in the thermally stabilized section of a channel in a laminar flow of non-newtonian fluid, relations (17) and (24) obtained for the Nu number may be approximated by the simpler expressions:

for a plane channel

$$\mathrm{Nu} = 9.84 - 2.9/(0.26n + 1); \tag{25}$$

for a circular tube

$$\mathrm{Nu} = 1/[0.373 - 0.2(n+1)/(n+3)]. \tag{26}$$

To illustrate the accuracy of the calculation, we shall compare values of the Nusselt number calculated according to (25) and (26), or, equivalently, according to (17) and (24), with exact theoretical solutions for the limiting cases n = 1 (newtonian fluid, parabolic velocity profile) and n = ∞ (piston flow) [3, 4] (see table).

It follows from the table that the method of solution developed and the formulas obtained allow calculation of the heat transfer with sufficient accuracy.

Values of Nusselt Number

Channel shape	Velocity profile	Nusselt number	
		exact solution	approximate solution
Plane	u = const	9.87	9.84
	parabola	7.56	7.54
Circular	u = const	5.78	5.78
	parabola	3.66	3.66

NOTATION

t—temperature of fluid; t_w—channel wall temperature; q_w—heat flux at channel wall; x—coordinate directed along the flow; h—coordinate directed across the flow, computed from center-line of plane channel; r—coordinate in the direction of the tube radius, calculated from tube axis; ξ—dimensionless coordinate; $2H$—channel height; $2R$—tube diameter; $\bar{u}$—mean fluid flow velocity; a—thermal diffusivity; λ—thermal conductivity.

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